TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 308, Number 1, July 1988

CONFORMAL DISTORTION OF BOUNDARY SETS

D. H. HAMILTON

ABSTRACT. Conformal maps f of the disk into itself have the property that $\dim f^{-1}(F) \leq \dim F$ for any set F on the unit circle.

1. Introduction. A common theme in complex analysis is the estimation of the Hausdorff dimension of special sets, for example the limit set of a Fuchsian group or the support of harmonic measure. A connecting link is the boundary distortion caused by conformal maps. For instance let f map the unit disk $\mathbf{D}$ conformally onto a fundamental region Ω of a Fuchsian group. A classical lemma of Löwner [**10**] shows that the length of $\overline{\Omega} \cap \partial\mathbf{D}$ is greater than $f^{-1}(\overline{\Omega} \cap \partial\mathbf{D})$; however if the latter is zero this is of no use for estimating Hausdorff dimension. We prove

THEOREM 1. *For any conformal map f on $\mathbf{D}$, with $f(\mathbf{D}) \subset \mathbf{D}$ and set $E \subset \partial\mathbf{D}$ with angular limits $f(E) \subset \partial\mathbf{D}$ we have*

$$\dim(f(E)) \geq \dim(E).$$

The result is provided not by a distortion theorem for Hausdorff measures but from an estimate for α-capacities. Recall that the inner α-capacity $C_\alpha(E)$ of a set E is defined by means of kernels

$$k_\alpha(t) = \begin{cases} \log(1/t), & \alpha = 0, \\ 1/t^\alpha, & 0 < \alpha \leq 1, \end{cases}$$

so that

$$C_\alpha(E) = k_\alpha^{-1}\left\{\inf_\mu \iint_{EE} k_\alpha(|x-y|)\,d\mu(x)\,d\mu(y)\right\},$$

where the infimum is taken over all probability measures μ supported by compact subsets of E (see Carleson [**5**]).

Now let $\mathscr{S}$ be the family of functions f one-to-one analytic on $\mathbf{D}$, satisfying $f(0) = 0$. Theorem 1 is an immediate consequence of

THEOREM 2. *Suppose that for $f \in \mathscr{S}$ we have $f(\mathbf{D}) \subset \mathbf{D}$ and some $E \subset \partial\mathbf{D}$ with angular limits $f(E) \subset \partial\mathbf{D}$. Then for $0 \leq \alpha \leq 1$*

$$C_\alpha(f(E)) \geq |f'(0)|^{-1/2} C_\alpha(E).$$

REMARKS. The case $\alpha = 0$ is due to Pommerenke (see [**18**, p. 348]).

Actually our results are somewhat more general than the case $f(\mathbf{D}) \subset \mathbf{D}$ and $f(E) \subset \partial\mathbf{D}$. Suppose that for a general Riemann mapping $f\colon \mathbf{D} \to \Omega$ there is a

Received by the editors July 8, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30C35; Secondary 30C55, 30C40.

Research supported in part by NSF Grant 8501509.

Dinismooth curve γ outside Ω so that $f(E) \subset \gamma$. By composition of conformal maps we deduce $\dim(f(E)) \geq \dim(E)$. This may be compared with a result of Markov [12] who proves for arbitrary f and $E \subset \partial\mathbf{D}$ $\dim(f(E)) \geq \psi(\dim E)$, where ψ is an increasing function with $\psi(t) \geq 2t/(1+t)$ as $t \to 1$. This sharpens a classical result of Beurling [3] that $\psi(t) \geq t/2$. Beurling also proved that $C_0(f(E)) = 0$ implies $C_0(E) = 0$. We extend Beurling's theory to general capacities.

DEFINITION 1. Let $h\colon \mathbf{R} \to \mathbf{R}$ be absolutely monotone, i.e., $h, h', h'', \dots$ are positive. Define kernel $k(t) = h(\log(1/t))$ and inner capacity

$$C_h(E) = k^{-1}\left\{\inf \iint_{EE} k(|x-y|)\, d\mu(x)\, d\mu(y)\right\},$$

where the infimum is taken over probability measures μ supported on compact subsets of E. We say that such a capacity is admissible if $\int_0^1 h(\log(1/t))t\, dt < \infty$. Carleson [5] studied such capacities in the case that h is monotone.

THEOREM 3. *Suppose that the capacities C_j $(j = 1, 2)$ derived from kernels $h(\log(1/t^j))$ are admissible. Then for any function $f(z) = \sum_{n=0}^{\infty} a_n z^{-n}$ univalent in $\{|z| > 1\}$ and for any $E \subset \partial\mathbf{D}$*

$$C_1(f(E)) \geq \{C_2(E)\}^{1/2}.$$

COROLLARY 1. *Suppose that h is strongly monotone with polynomial growth. Then for any f univalent on $\mathbf{D}$ and $E \subset \partial\mathbf{D}$ we have $C_h(E) = 0$ whenever $C_h(f(E)) = 0$.*

Beurling also studied the set $E \subset \partial\mathbf{D}$ on which a univalent function could be zero, showing that $C_0(E) = 0$. On the other hand Carleson [4] shows that sets of positive logarithmic capacity need not be sets of uniqueness for the Dirichlet class. In [1] it was asked if $C_0(E) > 0$ and $f_1 = f_2$ on E for $f_j \in \mathcal{S}$ implies $f_1 \equiv f_2$. We show that there is a set $E \subset \partial\mathbf{D}$ of dimension 1 which is not a set of uniqueness for $\mathcal{S}$. Here there is a strong connection between fixed sets of $\mathcal{S}$ and zero sets of Hölder classes, (see also Döppel, Köditz and Timman [6]). In particular E is the fixed set of an analytic flow if and only if E is a Carleson set, i.e. E has length zero and if its complementary subarcs have lengths l_n satisfying $\sum l_n \log(1/l_n) < \infty$.

2. Outline of the proof. The idea of the proof is easy to explain when we assume that the conformal map $f\colon \mathbf{D} \to \mathbf{D}$ has smooth extension to $\overline{\mathbf{D}}$.

First we use an inequality of Nehari (see §4) which holds for $f \in \mathcal{S}$ and $f(\mathbf{D}) \subset \mathbf{D}$. In the case that f is smooth on $\partial\mathbf{D}$ and $E \subset \partial\mathbf{D}$ is mapped into $\partial\mathbf{D}$ the Nehari inequality may be transformed to

$$\iint_{EE} \log\frac{1}{|f(x)-f(y)|}\, d\lambda(x)\, d\lambda(y) \leq \iint_{EE} \log\frac{1}{|x-y|}\, d\lambda(x)\, d\lambda(y)$$

for all real measures λ (supported on E) which are admissible, i.e.,

$$\iint_{EE} \log\frac{1}{|x-y|}\, d\lambda(x)\, d\lambda(y) < \infty.$$

Now as $E, f(E) \subseteq \partial\mathbf{D}$ the kernels are positive semidefinite so we may use a lemma of Schur (see §3) to "exponentiate" the inequalities. Thus we get for $0 < \alpha \leq 1$

$$\iint_{EE} \frac{d\lambda(x)\, d\lambda(y)}{|f(x)-f(y)|^{\alpha}} \leq \iint_{EE} \frac{d\lambda(x)\, d\lambda(y)}{|x-y|^{\alpha}}$$

for all admissible measures λ, i.e.,

$$\iint_{EE} \frac{d\lambda(x)\, d\lambda(y)}{|x-y|^\alpha} < \infty.$$

We then let λ be a sequence of probability measures such that

$$\iint_{EE} \frac{d\lambda(x)\, d\lambda(y)}{|x-y|^\alpha} \to \{C_\alpha(E)\}^{-\alpha}$$

and thus from the definition of α-capacity

$$\{C_\alpha(f(E))^{-\alpha} \leq \inf \iint_{EE} \frac{d\lambda(x)\, d\lambda(y)}{|f(x)-f(y)|^\alpha} \leq \{C_\alpha(E)\}^{-\alpha}.$$

Unfortunately this does not work very well in general. We need to approximate E by sets in $\mathbf{D}$ and then take care of the fact that our kernels are no longer positive semidefinite.

One might also expect to prove Theorem 1 from a result for Hausdorff measures. However it is not true that if $f \in \mathscr{S}$ maps $\mathbf{D}$ into $\mathbf{D}$ and $E \subset \partial\mathbf{D}$ into $\partial\mathbf{D}$ that $\Lambda(f(E)) \geq \Lambda(E)$ for any Hausdorff measure Λ. For instance there are f mapping sets E of positive logarithmic measure (but zero logarithmic capacity) to a single point. Consequently the method of Löwner's inequality will not work.

3. Quadratic inequalities. The method of exponentiating quadratic inequalities based on a lemma of Schur was introduced by Löwner [**10**] and FitzGerald [**7**]. Hamilton [**9**] introduced a continuous version for boundary value problems.

Let $A = (a_{ij})_{n\times n}$ be $n \times n$ complex matrices. The previous theory depends on two notions of Schur for A symmetric, B hermitian positive definite

(i) Schur product $AB = (a_{ij}b_{ij})_{n\times n}$,

(ii) Schur inequality $A < B$:

$$\left|\sum_{i=1}^n \sum_{j=1}^n a_{ij}\lambda_i\lambda_j\right| \leq \sum_{i=1}^n \sum_{j=1}^n b_{ij}\lambda_i\overline{\lambda}_j \tag{1}$$

for all $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$.

The fundamental result is $A < B \Rightarrow A^2 < B^2$. In this paper we have the slightly different situation of real symmetric matrices and the inequality

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij}\lambda_i\lambda_j \leq \sum_{i=1}^n \sum_{j=1}^n b_{ij}\lambda_i\lambda_j \tag{2}$$

for all real $\lambda_1, \ldots, \lambda_n$. Furthermore we know that A and B are positive semidefinite.

LEMMA 1. *For all real* $\lambda_1, \ldots, \lambda_n$ *and* $m = 1, 2, \ldots$

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij}^m\lambda_i\lambda_j \leq \sum_{i=1}^n \sum_{j=1}^n b_{ij}^m\lambda_i\lambda_j.$$

This is easily proved by induction noting that the Schur product is commutative: assuming $B^{m-1} - A^{m-1} > 0$ and multiplying by $A > 0$ Schur's lemma for products gives $AB^{m-1} - A^m > 0$. On the other hand $B - A > 0$ and multiplying by B^{m-1} gives $B^m - B^{m-1}A > 0$. Adding these two gives $B^m - A^m > 0$, which proves the lemma.

LEMMA 2. *For any absolutely monotone function $h(t)$:*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} h(a_{ij})\lambda_i\lambda_j \le \sum_{i=1}^{n}\sum_{j=1}^{n} h(b_{ij})\lambda_i\lambda_j. \tag{3}$$

for all $\lambda_1, \dots, \lambda_n \in \mathbf{R}$.

This lemma follows by induction from Lemma 1 for all polynomials $h(t)$ with positive coefficients. Thus by approximation it holds for $h(t) = e^{\alpha t}$, $\alpha \ge 0$. Finally it is well known that for any absolutely monotone h

$$h(t) = \int_0^\infty e^{\alpha t}\, d\mu(\alpha)$$

for some positive measure μ, which together with the above consideration proves (3).

Actually we need to consider continuous versions of (3).

LEMMA 3. *Suppose that A, B are real symmetric continuous on $\mathbf{D}^2$ and for any real-valued continuous function $\lambda(x)$ having compact support on $\mathbf{D}$ we have*

$$\begin{cases} \iint A(x,y)\lambda(x)\lambda(y)|dx|^2|dy|^2 \ge 0, \\ \iint B(x,y)\lambda(x)\lambda(y)|dx|^2|dy|^2 \ge 0, \end{cases} \tag{4}$$

$$\iint B(x,y)\lambda(x)\lambda(y)|dx|^2|dy|^2 \ge \iint A(x,y)\lambda(x)\lambda(y)|dx|^2|dy|^2. \tag{5}$$

Then for any absolutely monotone function $h(t)$

$$\iint h(B(x,y))\lambda(x)\lambda(y)|dx|^2|dy|^2 \ge \iint h(A(x,y))\lambda(x)\lambda(y)|dx|^2|dy|^2 \tag{6}$$

for all real-valued continuous $\lambda(x)$ with compact support.

This is proved from Lemma 2 by approximating with Riemann sums (see Hamilton [**9**]). Further approximation arguments show that we may allow A, B to have singularities and λ to be a suitable measure. Carleson [**5**] considers kernels with singularities of the form $k(t) = \log 1/t$ and establishes potential theory for kernels $h(k)$. Carleson requires that $\int_0^1 h(\log 1/t)t\, dt < \infty$. Consequently we have a class of admissible real measures μ satisfying

$$\iint_{\mathbf{DD}} h\left(\log\frac{1}{|x-y|}\right)|d\lambda(x)||d\lambda(y)| < \infty,$$

i.e. the potential $\int h(\log(1/|x-y|))\, d\lambda(x)$ has finite energy. Putting these facts together we get

PROPOSITION 1. *Suppose that A, B are real symmetric on $\mathbf{D}^2$, continuous except on the diagonal. Let*

$$|A(x,y)| + |B(x,y)| \le C\log\frac{1}{|x-y|}.$$

Then if (4), (5) *are satisfied and h is any absolutely monotone function such that $h(\log(1/|z|)) \in L^1(\mathbf{R}^2)$ we have*

$$\iint h(B(x,y))\, d\lambda(x)\, d\lambda(y) \ge \iint h(A(x,y))\, d\lambda(x)\, d\lambda(y)$$

for all real admissible measures supported on compact subsets of $\mathbf{D}$.

4. Golusin and Nehari inequalities. The following two quadratic inequalities are essential.

THEOREM A (GOLUSIN [17]). *Suppose that $f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}$ is univalent in $\Omega = \{|z| > 1\}$. Then for any points $z \in \Omega$ $(j = 1, \dots, n)$ and complex numbers $\lambda_1, \dots, \lambda_n$*

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j \log\left|\frac{f(z_i) - (z_j)}{z_i - z_j}\right|\right| \le \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \overline{\lambda}_j \log\frac{1}{(1 - 1/z_i\overline{z}_j)}.$$

THEOREM B (NEHARI [13]). *Suppose that $f \in \mathscr{S}$, $f(\mathbf{D}) \subset \mathbf{D}$. Then*

$$0 \le \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_k \lambda_j \log\left|\frac{f'(0) z_j z_k}{f(z_j) f(z_k)} \frac{f(z_j) - f(z_k)}{z_j - z_k} \frac{1 - f(z_j)\overline{f(z_k)}}{1 - z_j \overline{z}_k}\right|$$

for all $z_n, \dots, z_n \in \mathbf{D}$ and $\lambda_1, \dots, \lambda_n \in \mathbf{R}$.

The proof of continuous versions is immediate.

LEMMA 4. *For $f \in \mathscr{S}$, $f(\mathbf{D}) \subset \mathbf{D}$*

$$\iint_{\mathbf{DD}} \log\frac{|f'(0)xy|}{|(x - y)(1 - x\overline{y})|} \lambda(x)\lambda(y)|dx|^2|dy|^2$$
$$\ge \iint_{\mathbf{DD}} \log\frac{|f(x)f(y)|}{|(f(x) - f(y))(1 - f(x)\overline{f(y)})|} \lambda(x)\lambda(y)|dx|^2|dy|^2$$

for all real continuous λ supported on a compact subset of $\mathbf{D}$.

The quadratic inequality of Lemma B may be rearranged to give

$$\sum\sum_{j \ne k} \lambda_j \lambda_k \log\frac{|f(z_j) f(z_k)|}{|(f(z_j) - f(z_k))(1 - f(z_j)\overline{f(z_k)})|}$$
$$\le \sum\sum_{j \ne k} \lambda_j \lambda_k \log\frac{|f'(0) z_j z_k|}{|(z_j - z_k)(1 - z_j \overline{z_k})|}$$
$$+ \sum_{j=1}^{n} \lambda_j^2 \log\left|\frac{f'(z_j)(1 - |f(z_j)|^2)}{(1 - |z_j|^2)}\right|,$$

which on keeping z_j in a fixed compact subset of $\mathbf{D}$ and using an approximation argument we can ensure that the diagonal sum tends to zero as $n \to \infty$. A similar argument yields

LEMMA 5. *Suppose that $f(z) = z \sum_{n=1}^{\infty} a_n z^{-n}$ is univalent in $\Omega = \{|z| > 1\}$. Then for any complex continuous λ compactly supported on Ω*

$$\left|\iint_{\Omega\Omega} \lambda(x)\lambda(y) \log\frac{f(x) - f(y)}{x - y} |dx|^2|dy|^2\right|$$
$$\le \iint_{\Omega\Omega} \lambda(x)\overline{\lambda(y)} \log\frac{1}{(1 - 1/x\overline{y})} |dx|^2|dy|^2.$$

Unfortunately the kernels of Lemma 5 are not (quite) positive semidefinite. Consequently we need to restrict our attention to a subset of $\mathbf{D}$ where the kernels are "nearly" positive semidefinite.

5. Proof of Theorems 1 and 2. Without loss of generality we restrict our attention to compact sets $E \subset \partial\mathbf{D}$, with $C_h(E) > 0$. In particular $C_0(E) > 0$. For any $\varepsilon > 0$ choose an open set $\widetilde{E}$ which is the union of Stolz angles with vertices on E such that $\widetilde{E}$ and $f\widetilde{E}$ are subsets of the annulus $\{1 - \varepsilon < |z| < 1\}$ and there is a compact $F \subset \widetilde{E}$ with

$$|C_h(E) - C_h(F)| < \delta,$$

and provided $C_h(f(\widetilde{E})) < \infty$,

$$|C_h(f(E)) - C_h(f(F))| < \delta.$$

First we note

LEMMA 6. *The kernel* $\log(1/|1 - \overline{x}y|)$ *is positive semidefinite on* $\mathbf{D}$.

We write

$$\int_{\mathbf{D}} \log \frac{1}{(1 - \overline{x}y)}\, d\mu(y) = \sum_{n=1}^{\infty} \frac{\mu_n}{n} \overline{x}^n,$$

where $\mu_n = \int y^n\, d\mu(y)$. Hence as μ is real

$$\iint_{\mathbf{DD}} \log \frac{1}{(1 - \overline{x}y)}\, d\mu(x)\, d\mu(y) = \sum_{n=1}^{\infty} \frac{|\mu_n|^2}{n}$$

for which we take real parts to get the required result.

Also we prove

LEMMA 7. *The kernel* $\log|(1 - x\overline{y})/(x - y)|$ *is positive semidefinite on* $\mathbf{D}$.

Let us define $G(x, y) = \log|(1 - \overline{x}y)/(x - y)|$, i.e., the Green's function of $\mathbf{D}$. Thus by Riesz representation of harmonic functions (see [**5**]) if we define

$$u(x) = \int_{\mathbf{D}} G(x, y)\, d\mu(y),$$

provided $d\mu = \mu(y)|dy|^2$ where μ is C^2 then $\Delta u = (-2\pi)\mu$ and $u = 0$ on $\partial\mathbf{D}$. Consequently from Green's formulae

$$\begin{aligned}
&\iint_{\mathbf{DD}} \log\left|\frac{1 - x\overline{y}}{x - y}\right| d\mu(x)\, d\mu(y) \\
&\quad = (-2\pi) \int_{\mathbf{D}} u\Delta u |dx|^2 \\
&\quad = (2\pi) \int_{\mathbf{D}} |\nabla u|^2 |dx|^2 > 0,
\end{aligned}$$

which proves the lemma.

Thus combining these results

LEMMA 8. *The kernel* $\log(1/|(x - y)(1 - \overline{x}y)|)$ *is positive semidefinite on* $\mathbf{D}$.

This follows from writing the kernel as

$$2\log\frac{1}{|1 - \overline{x}y|} + \log\left|\frac{1 - x\overline{y}}{x - y}\right|$$

which are positive kernels by the previous lemma.

Finally we deal with the $\log|xy|$ terms. We restrict our attention to a thin annulus A: $\{1-\delta \le |x|, |y| \le 1\}$. Now (even on A) $\log|xy|$ has positive and negative eigenvalues, so we find a kernel $K(x,y,\varepsilon)$ on A such that $\log|xy| + K(x,y,\varepsilon)$ and $\log|f(x)f(y)| + K(x,y,\varepsilon)$ are positive semidefinite on A; also we require that

$$\lim_{\varepsilon\to 0} \sup_A |K(x,y,\varepsilon)| = 0.$$

To do this we use spectral resolution of the real symmetric kernel $\log|xy|$. On $L^2(A)$ the operator $\mathscr{L}g = \int_A \{\log|xy|\} g(y)|dy|^2$ spans a space with basis 1, $\log|x|$. Thus $\mathscr{L}$ has rank 2 and

$$\log|xy| = \lambda_1\phi_1(x)\phi_1(y) + \lambda_2\phi_2(x)\phi_2(y)$$

where λ_j, ϕ_j are eigenvalues and orthonormal eigenfunctions of $\mathscr{L}$. Suppose that $\lambda_1 > 0$ and $\lambda_2 < 0$. Then we set

$$K(x,y,\varepsilon) = -\lambda_2\phi_2(x)\phi_2(y)$$

and we shall see later that it has the required properties. First we need to estimate the λ_j. It is clear that we need only consider the effect of $\mathscr{L}$ on functions of the form

$$g(y) = \alpha + \beta\log|y|.$$

Then

$$\mathscr{L}g = (\tau_1\alpha + \tau_2\beta) + (\tau_0\alpha + \tau_1\beta)\log|x|$$

where

$$\tau_0 = \int_A 1\cdot|dy|^2, \quad \tau_1 = \int_A \log|y||dy|^2$$

and

$$\tau_2 = \int_A (\log|y|)^2|dy|^2.$$

Consequently the equation for eigenvalues is

$$\begin{bmatrix}\tau_1 & \tau_2\\ \tau_0 & \tau_1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \lambda\begin{bmatrix}\alpha\\ \beta\end{bmatrix}.$$

First we estimate the τ_j to $O(\varepsilon^5)$; a simple computation yields

LEMMA 9. *Under the above notation*

$$\tau_0 = 2\pi\left(\varepsilon - \tfrac{1}{2}\varepsilon^2\right),$$
$$\tau_1 = 2\pi\left\{-\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{6} + \frac{\varepsilon^4}{24} + O(\varepsilon^5)\right\},$$
$$\tau_2 = 2\pi\frac{\varepsilon^3}{3} + O(\varepsilon^5).$$

A direct computation now yields the eigenvalues $(j = 1,2)$

$$\begin{aligned}\lambda_j &= \tau_1 + (-1)^j\sqrt{\tau_0\tau_2}\\ &= 2\pi\left\{\left(-\frac{1}{2} + \frac{(-1)^j}{\sqrt{3}}\right)\varepsilon^2 + \left(\frac{1}{6} + \left(-\frac{(-1)^j}{4\sqrt{3}}\right)\right)\varepsilon^2 + \cdots\right\}.\end{aligned}$$

Next we compute the eigenfunctions

$$\psi_j = \alpha_j + \beta_j \log|x|.$$

For $j = 1$ we solve

$$\begin{bmatrix} \left(\frac{1}{\sqrt{3}}\varepsilon^2 + \cdots\right) & \frac{\varepsilon^3}{3} + \cdots \\ \varepsilon - & \left(\frac{1}{\sqrt{3}}\varepsilon^2 + \cdots\right) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we can solve to get

$$\psi_1 = \left(-\sqrt{3}\varepsilon\right) + \log|y|.$$

Then

$$\begin{aligned} \int_A |\psi_1|^2 |dy|^2 &= 3\varepsilon^2\tau_0 - 2\sqrt{3}\varepsilon\tau_1 + \tau_2 \\ &= 2\pi\varepsilon^3\left(3 + \sqrt{3} + \tfrac{1}{3} + \cdots\right). \end{aligned}$$

Consequently the normalized eigenfunction is

$$\phi_1 = \frac{-\varepsilon^{-1/2}}{\sqrt{2\pi(1 + 1/\sqrt{3} + \frac{1}{9})}} + \frac{-\varepsilon^{-3/2}\log|y|}{\sqrt{2\pi(3 + \sqrt{3} + \frac{1}{3})}}.$$

In particular we have, for $x, y \in A$, $\lambda_1\phi_1(x)\phi_1(y) = O(\varepsilon^2\varepsilon^{-1}) = O(\varepsilon)$ as $\log|x| = O(\varepsilon)$ on A. We then obtain a similar result for the other eigenvalue. Let us summarize this in

LEMMA 10. *As* $\varepsilon \to 0$; $x, y \in \{1 - \varepsilon < |z| < 1\}$.

$$\lambda_j\phi_j(x)\phi_j(y) = O(\varepsilon).$$

In particular we see that the function $K(x, y, \varepsilon)$ is $O(\varepsilon)$.

LEMMA 11. *The following kernels are positive semidefinite on* $\{1-\varepsilon < |x|, |y| < 1\}$:

$$K_1 = \log|xy| + K(x, y, \varepsilon), \quad K_2 = K(x, y, \varepsilon).$$

This is because, by explicit construction, the eigenvalue of K_j is $(-1)^{j+1}\lambda_j$. Next we consider the kernel $\log|f(x)f(y)|$. It would be most complicated to follow through the above procedure; however by change of variables we can easily reduce to the above case. Also we need to observe that we are restricting ourselves to $x, y \in \widetilde{E}$, $f(\widetilde{E}) \subset \{1 - \varepsilon < |z| < 1\}$.

LEMMA 12. *The following kernels are positive semidefinite on* $\widetilde{E}$:

$$\begin{aligned} K_3 &= \log|f(x)f(y)| + K(f(x), f(y), \varepsilon), \\ K_4 &= K(f(x), f(y), \varepsilon). \end{aligned}$$

To prove this we observe that

$$K_3 = K_1(f(x), f(y)) = \lambda_1\phi_1(f(x))\phi_1(f(y))$$

and

$$K_4 = \lambda_2\phi_2(f(x))\phi_2(f(y))$$

and each of these is positive semidefinite.

REMARKS. If K_1, K_2 had contained more than one eigenfunction the above substitution would not have worked. Finally we have, analogous to Lemma 10, LEMMA 13. *As* $\varepsilon \to 0$, $x, y \in \widetilde{E}$ $K_j(x,y) = O(\varepsilon)$.

We are now ready to prove Theorem 1. We add

$$\iint (K_2(x,y) + K_4(x,y))\lambda(x)\lambda(y)|dx|^2|dy|^2$$

to both sides of the quadratic inequality of Lemma 4, to obtain for admissible λ:

$$\begin{aligned}\iint_{\widetilde{E}\widetilde{E}} A(x,y)\lambda(x)\lambda(y)|dx|^2|dy|^2 \\ &\le \iint_{\widetilde{E}\widetilde{E}} (B(x,y) + \log|f'(0)|)\lambda(x)\lambda(y)|dx|^2|dy|^2 \\ &\le \iint_{\widetilde{E}\widetilde{E}} B(x,y)\lambda(x)\lambda(y)|dx|^2|dy|^2,\end{aligned}$$

as $|f'(0)| \le 1$. Now

$$A(x,y) = \log\frac{|f(x)f(y)|}{|(f(x)-f(y))(1-f(x)\overline{f(y)})|} + K_2 + K_4$$

and

$$B(x,y) = \log\frac{|xy|}{|(x-y)(1-\overline{x}y)|} + K_2 + K_4.$$

By our previous lemmas, A and B are positive semidefinite. Thus we may apply Proposition 1, i.e., for any absolutely monotone h with $h(\log(1/|z|)) \in L^1(\mathbf{R}^2)$, letting $h_0(t) = h(t/2)$

$$\iint h_0(A(x,y))\,d\lambda(x)\,d\lambda(y) \le \iint h_0(B(x,y))\,d\lambda(x)\,d\lambda(y)$$

for all admissible real measures $d\lambda$. Next we note that for any $\delta > 0$, there is an $\varepsilon > 0$ and an $\widetilde{E}$ such that

$$h(A(x,y)) = h_0\left(\log\frac{1}{|f(x)-f(y)|^2}\right) + O(\delta),$$
$$h(B(x,y)) = h_0\left(\log\frac{1}{|x-y|^2}\right) + O(\delta)$$

uniformly on $\widetilde{E}$. This follows from Lemma 13 and the definition of $\widetilde{E}$. Consequently we have

$$\begin{aligned}\iint_{\widetilde{E}\widetilde{E}} h\left(\log\frac{1}{|f(x)-f(y)|}\right)\,d\lambda(x)\,d\lambda(y) \\ &\le \iint_{\widetilde{E}\widetilde{E}} h\left(\log\frac{1}{|x-y|}\right)\,d\lambda(x)\,d\lambda(y) + O(\delta)\end{aligned}$$

for any measure μ with

$$\iint_{\widetilde{E}\widetilde{E}} h\left(\log\frac{1}{|x-y|}\right)\,d\lambda(x)\,d\lambda(y)$$

finite. We now let our measures be probability measures supported by F. In particular let λ_n be a sequence such that

$$\iint_{\widetilde{F}\widetilde{F}} h\left(\log\frac{1}{|x-y|}\right)\,d\lambda_n(x)\,d\lambda_n(y)$$

tends toward the infimum. Thus by the above inequality

$$\inf_\lambda \iint_{f(F)f(F)} h\left(\log\frac{1}{|x-y|}\right) d\lambda(x)\,d\lambda(y) \le \inf_\lambda \iint_{FF} h\left(\log\frac{1}{|x-y|}\right) d\lambda(x)\,d\lambda(y) + \delta.$$

Consequently recalling the construction of F, letting $\delta \to 0$ and taking $e^{h^{-1}}$ of both sides gives us a generalization of Theorem 2:

PROPOSITION 2. *Suppose that $f \in \mathscr{S}$, $f(\mathbf{D}) \subset \mathbf{D}$ and $E \subset \partial\mathbf{D}$ with $f(E) \subset \partial\mathbf{D}$. Then for any admissible capacity*

$$C_h(f(E)) \ge C_h(E).$$

This is not quite a generalization of Theorem 1 because we lost our $|f'(0)|$ factor. To prove Theorem 1 we begin with

$$\iint_{\widetilde{E}\widetilde{E}} A_1(x,y)\,d\lambda(x)\,d\lambda(y) \le \iint_{\widetilde{E}\widetilde{E}} B(x,y)\,d\lambda(x)\,d\lambda(y)$$

where $A_1 = A(x,y) + \log(1/|f'(0)|)$ which positive semidefinite. We then follow through the previous method with explicit function $h(t) = e^{\alpha t}$ and pick up the extra factor.

Theorem 1 follows immediately from Frostman's relation between capacity and Hausdorff measure (see [**5**]).

6. Proof of Theorem 3. The argument here is similar to the previous section so we just sketch it. We use the continuous form of the Golusin inequalty to write

$$\left|\iint \lambda(x)\lambda(y)\log\frac{1}{|f(x)-f(y)|}|dx|^2|dy|^2\right| \le \iint \lambda(x)\lambda(y)\log\frac{1}{|1-1/x\overline{y}|}|dx|^2|dy|^2 + \left|\iint \lambda(x)\lambda(y)\log\frac{1}{|x-y|}|dx|^2|dy|^2\right|$$

for all real-valued continuous λ compactly supported on Ω. Then as in §2 we restrict ourselves to λ supported on $\{1 < |z| < 1+\varepsilon\}$.

From an approximation argument similar to that of §2

$$\log\frac{1}{|x-y|} = \log\frac{1}{|1-1/x\overline{y}|} + A(x,y,\varepsilon),$$

where $A(x,y,\varepsilon) \to 0$ uniformly as $\varepsilon \to 0$. Thus we get

$$\left|\iint \lambda(x)\,\lambda(y)\log\frac{1}{|f(x)-f(y)|}|dx|^2|dy|^2\right| \le 2\iint \lambda(x)\lambda(y)\log\frac{1}{|1-1/x\overline{y}|}|dx|^2|dy|^2 + \left|\iint \lambda(x)\lambda(y)A(x,y,\varepsilon)|dx|^2|dy|^2\right|$$

where $A(x,y,\varepsilon)$ uniformly tends to 0 as $\varepsilon \to 0$, and is positive semidefinite. Now any such quadratic inequality for real kernels implies the same inequality with complex $\lambda(x)$ supported in $\{1 < |z| < 1+\varepsilon\}$, see FitzGerald and Horn [**8**]. Consequently for any complex-valued continuous $\lambda(x)$ compactly supported on $\{1 < |z| < 1+\varepsilon\}$:

$$\left|\iint \lambda(x)\lambda(y)\log\frac{1}{|f(x)-f(y)|}|dx|^2|dy|^2\right| \\ \leq \iint \lambda(x)\overline{\lambda}(y)\left\{\log\frac{1}{|1-1/x\overline{y}|^2}+A(x,y)\right\}|dx|^2|dy|^2.$$

We may then "exponentiate" this expression in accordance with Schur's lemma. Thus for any absolutely monotone function h

$$\left|\iint \lambda(x)\lambda(y)h\left\{\log\frac{1}{|f(x)-f(y)|}\right\}|dx|^2|dy|^2\right| \\ \leq \iint \lambda(x)\lambda(y)h\left\{\log\frac{1}{|1-1/x\overline{y}|^2}\right\}|dx|^2|dy|^2+O(\varepsilon),$$

for all continuous positive $\lambda(x)$ supported in $\{1 < |z| < 1+\varepsilon\}$, and satisfying $\int \lambda(x)|dx|^2 = 1$. To complete the proof of Theorem 2 we take sets $\widetilde{E} \subset \{1 < |z| < 1+\varepsilon\}$ such that $C_h(E)$ is approximated by $C_h(\widetilde{E})$, and $C_h(f(E))$ by $C_h(f(\widetilde{E}))$. Thus the proof in this section is just a slight variation of that in the previous section.

Finally we prove the Corollary. Notice that if h has slow growth then for any set E:

$$C_2(E) \geq C_1(E)^n.$$

To see this one simply inspects the kernels. Consequently by Theorem 3 $C_h(f(E)) = 0$ implies $C_h(E) = 0$.

7. Fixed points of conformal maps. Carleson sets are exactly the zero sets of the class A_α of functions $h(z)$ analytic on $\mathbf{D}$ and Hölder continuous with exponent α, $0 < \alpha \leq 1$, on $\overline{\mathbf{D}}$, see [**4**].

Now we define the concept of a holomorphic conformal flow. We say that a class of functions $f_\lambda \in \mathscr{S}$, with parameter $\{|\lambda| < 1\}$ is a holomorphic flow if

(i) $f_\lambda(z)$ is holomorphic in $\lambda \in \mathbf{D}$ for fixed z.

(ii) $f_0(z) \equiv z$.

The importance of holomorphic flows comes from quasiconformal theory. A set $E \subset \partial\mathbf{D}$ is a fixed set for a holomorphic flow f_λ if $f_\lambda(z) = z$ for $z \in E$, $|\lambda| < 1$.

PROPOSITION 3. *A set $E \subset \partial\mathbf{D}$ of positive capacity is a fixed set for a holomorphic conformal flow if and only if E is a Carleson set.*

Now if E is a Carleson set there is a nontrivial $h \in A_1$ with $h(0) = 0$, $\|h'\|_\infty < 1$ and $h(z) = 0$ on E. Then we set

$$f_\lambda(z) = z + \lambda h(z)$$

and note that for $|\lambda| < 1$, $z_j \in \mathbf{D}$

$$|f_\lambda(z_1) - f_\lambda(z_2)| \geq |z_1 - z_2|(1 - |\lambda|\|h'\|_\infty),$$

which implies that $f_\lambda \in \mathscr{S}$.

Conversely, if f_λ is a holomorphic conformal flow then by the λ-lemma of Bers and Royden [**2**], $f_\lambda(z)$ has a quasiconformal extension to $\mathbf{C}$ for $|\lambda| < 1$. Consequently, E is a zero set of the Hölder continuous function $f_\lambda(z) - z$. Therefore E is a Carleson set.

The theorem shows that any Carleson set is a set of nonuniqueness for $\mathcal{S}$. We now display a Carleson set of dimension 1. It is easier to think of $\partial\mathbf{D}$ with normalized arc length as the interval $(0,1)$. We construct a Cantor set. At the first stage, remove an open interval I_1 of length $(1-\alpha_1)$, etc., so that at the nth stage we get a set E_n consisting of 2^n subarcs of length $\alpha_1 \cdots \alpha_{n-1}(1-\alpha_n)/2^n$. To see that E is a Carleson set, note $|E| \le \alpha_1 \cdots \alpha_n$ and

$$\sum_j l_{j,n} \log(l_{j,n}^{-1}) = \sum_{n=1}^{\infty} (\alpha_1 \cdots \alpha_{n-1})(1-\alpha_n) \log \frac{2^n}{(\alpha_1 \cdots \alpha_{n-1})(1-\alpha_n)}.$$

Let us choose $\alpha_n = 1 - n^{-1/2}$. Then as $n \to \infty$

$$\alpha_1 \cdots \alpha_n \to 0$$

while

$$\sum_j l_{j,n} \log(l_{j,n}^{-1}) \le C_1 \sum (n-1) e^{c_2 - n^{1/2}} < \infty.$$

On the other hand E has dimension 1 since if the α_j are constant $\alpha < 1$, $\dim E = \log 2/\log \frac{2}{\alpha}$. However the Hausdorff dimension of E is not affected by finitely many α_j. Thus as $\alpha_n \to 1$ we get $\dim E = 1$.

If we have $f \in \mathcal{S}$, with $f(\mathbf{D}) \subset \mathbf{D}$ then Theorem 1 says that any fixed set must have zero capacity. We now use the Löwner differential equation to generate fixed sets $E \subset \partial\mathbf{D}$ of flows $f_t \in \mathcal{S}$, $0 \le t \le 1$, and $f_t(\mathbf{D}) \subset \mathbf{D}$.

PROPOSITION 4. *Let $p(z)$ be analytic on $\mathbf{D}$, and*
(i) $p(z) = 0$, $z \in E$,
(ii) $p(0) = 1$, $\operatorname{Re} p > 0$ *on* $\mathbf{D}$,
(iii) $p(z)$ *has modulus of continuity* $\omega(t)$ *on* $\overline{\mathbf{D}}$ *satisfying*

$$\int_0^1 \frac{dt}{\omega(t)} = \infty.$$

Then the solution of the system $\dot{f}_t = -f_t p(f_t)$, $f_0(z) = z$ *is univalent on* $\mathbf{D}$, *continuously differentiable with respect to t on* $\overline{\mathbf{D}}$, *and satisfies* $f_t(z) = z$ *for* $z \in E$.

We do not include the complete details of the proof since it is only a variation of the standard one for the Löwner equation, see [**18**]. The existence and uniqueness of solutions of first-order systems can be proved under assumption (iii), the so-called Osgood condition. The uniqueness of solutions implies f_t is one-to-one on $\overline{\mathbf{D}}$.

Finally we observe that Propositions 3 and 4 actually have an application to "peak sets" of Hölder classes. Recall that $E \subset \partial\mathbf{D}$ is a peak set of a function h analytic on $\mathbf{D}$, continuous on $\overline{\mathbf{D}}$, if $\operatorname{Re} h > 0$ on $\mathbf{D}$ but $h(z) = 0$ on E. B. A. Taylor and D. Williams [**18**] prove that the peak set of the Lipschitz class A_1 is a finite set, while Noell and Wolff [**15**] prove that every peak set of A_α has $(1-\alpha)$ Hausdorff measure zero. These peak sets are closely related to the fixed points of Proposition 4. Observe that E is a peak set of p implies E is a fixed set for f_t. We define a general class of functions A_ω as follows. A_ω consists of functions h analytic on $\mathbf{D}$ with modulus of continuity $w(t)$. Now if $\int_0^1 dt/\omega(t) = \infty$ then A_ω falls between A_1 and $\bigcap_{\alpha<1} A_\alpha$. Thus by Proposition 4 any peak set of A_ω is a fixed set for S, with $f(\mathbf{D}) \subset \mathbf{D}$. Consequently from Theorem 1 we deduce

COROLLARY 2. *Let E be a peak set of A_ω, where $\int_0^1 dt/\omega = \infty$. Then $C_0(E) = 0$.*

REFERENCES

1. K. Barth, D. Brannan and W. K. Hayman, *Research problems in complex analysis*, Bull. London Math. Soc. **16** (1984), 490–517.
2. L. Bers and H. Royden, *Holomorphic families of injections*, Acta Math. **157** (1986), 259–286.
3. A. Beurling, *Ensembles exceptionelles*, Acta Math. **72** (1940), 1–13.
4. L. Carleson, *Sets of uniqueness for functions analytic in the unit disk*, Acta Math. **87** (1952), 325–345.
5. ——, *Selected problems on exceptional sets*, Van Nostrand, 1967.
6. K. Döppell, H. Köditz and S. Timman, *Bemurkungen über Fixpunktmengen schlichter Functionen*, Rend. Inst. Math. Univ. Trieste **8** (1976), 162–166 (1977).
7. C. H. FitzGerald, *Quadratic inequalities and coefficient estimates for schlicht functions*, Arch. Rational Mech. Anal. **46**, 356–368.
8. C. H. FitzGerald and R. Horn, *On the structure of hermitian symmetric matrices*, J. London Math. Soc. (2) **15** (1977), 419–430.
9. D. H. Hamilton, *Quadratic inequalities and interpolation*, J. Analyse Math. **43** (1984), 26–50.
10. C. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, Math. Ann. **89**, 103–121.
11. N. G. Makarov, *On the distortion of boundary sets under conformal mappings*, Proc. London Math. Soc. (3) **51** (1985), 369–384.
12. ——, *Conformal mapping and Hausdorff measures*, (preprint).
13. Z. Nehari, *Some inequalities in the theory of functions*, Trans. Amer. Math. Soc. **75** (1953), 256–286.
14. R. Nevanlinna, *Analytic functions*, Springer-Verlag, 1970.
15. A. V. Noell and T. H. Wolff, *On peak sets for* Lip α *classes*, J. Functional Anal. (to appear).
16. C. Pommerenke, *On the logarithmic capacity and conformal mapping*, Duke Math. J. **35** (1968), 321–326.
17. ——, *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
18. B. H. Taylor and D. L. Williams, *The peak sets of A^m*, Proc. Amer. Math. Soc. **24** (1970), 604–606.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742